MATH 1A - MIDTERM 3

PEYAM RYAN TABRIZIAN

Name: ____________________

Instructions: This midterm counts for 20% of your grade. You officially have 110 minutes to take this exam (although I will try to give you more time). This is a fairly long exam, so don't spend too much time on each question! May your luck be maximized :)

Note: Please check the following box if it applies to you:

☐ Today is the last day to change your grade to P/NP and vice-versa. Please check this box if your decision of changing your grade to P/NP depends on the score you'll receive on this exam, and you would like to have this exam graded by 5 pm (**please be honest**)

1		25
2		10
3		15
4		10
5		10
6		10
7		20
Bonus 1		5
Bonus 2		5
Total		100

Date: Friday, July 29th, 2011.

1. *(25 points)* Sketch a graph of the function $f(x) = x\ln(x) - x$. Your work should include:
   - Domain
   - Intercepts
   - Symmetry
   - Asymptotes (no Slant asymptotes, though)
   - Intervals of increase/decrease/local max/min
   - Concavity and inflection points

Note: You may use the axes provided on page 4 to draw your graph! Make sure to label all important points!

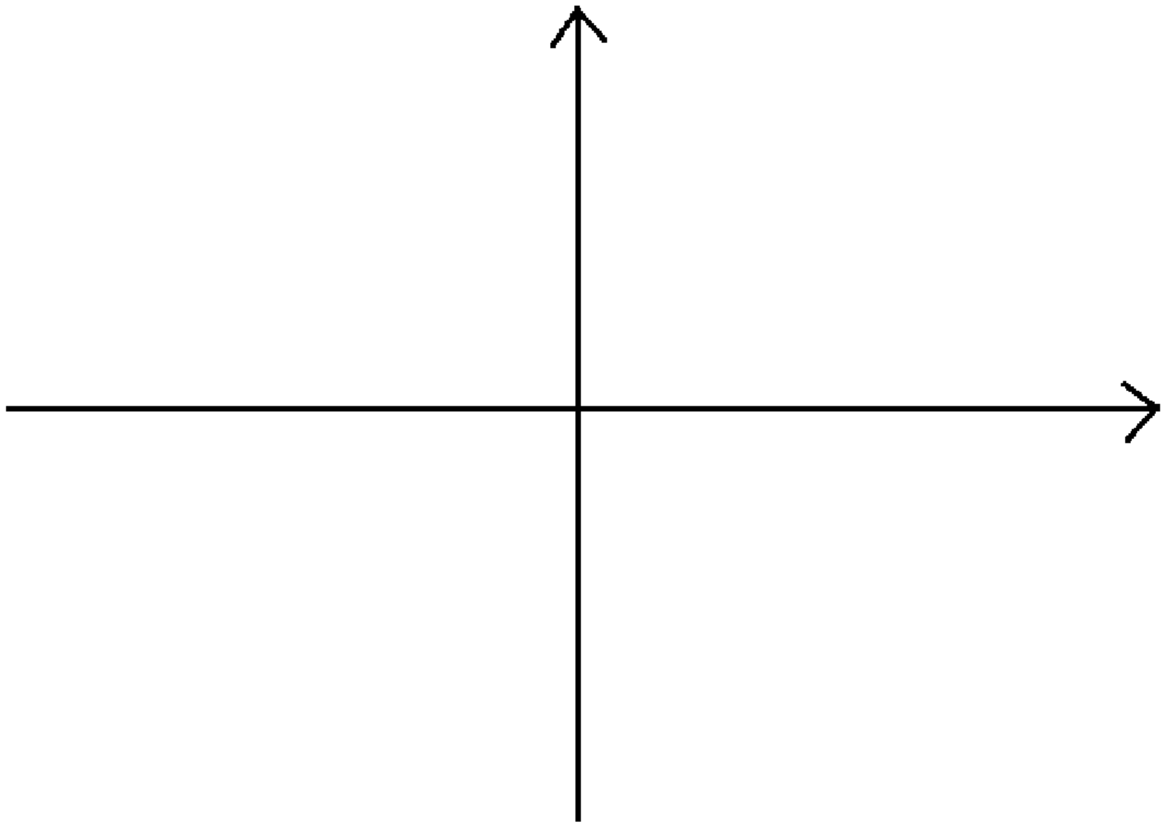
Hint: For the Asymptotes-part, it might help to notice that

$$f(x) = x(\ln(x) - 1)$$

Hint: There should be a nice simplification when you calculate $f'(x)$. If there's no simplification, then you made a differentiation mistake!

(This page is left blank in case you need more space to work on question 1.)

1A/Math 1A Summer/Exams/Axes.png



2. *(10 points)* Use a linear approximation (or differentials) to find an approximate value of $\sqrt{99}$

3. *(15 points)* Assume the radius of a cone is increasing at a rate of 3 cm/s while its height is decreasing at a rate of 1 cm/s. At what rate is its volume increasing/decreasing when its radius is 2 cm and its volume is $\frac{4\pi}{3}\ cm^3$?

   Note: If you don't remember the formula for the volume of a cone, do this problem with a cylinder instead (14 points max). If you're still stuck, do it with a triangle instead (11 points max).

4. *(10 points)* Find the following limits:

(a) $\lim_{x\to 0} x^2 \ln(x)$

(b) $\lim_{x\to\infty} x^{\frac{1}{x}}$

5. *(10 points)* Find the absolute maximum and minimum of f on $[0, 2]$, where:

$$f(x) = x^4 - 4x + 1$$

6. *(10 points)* Show that if $f'(x) > 0$ for all x, then f is increasing.

Hint: Assume $b > a$ and show that $f(b) > f(a)$

7. *(20 points)* Find the dimensions of the rectangle of largest area that can be inscribed in (put inside of) a circle of radius 1.

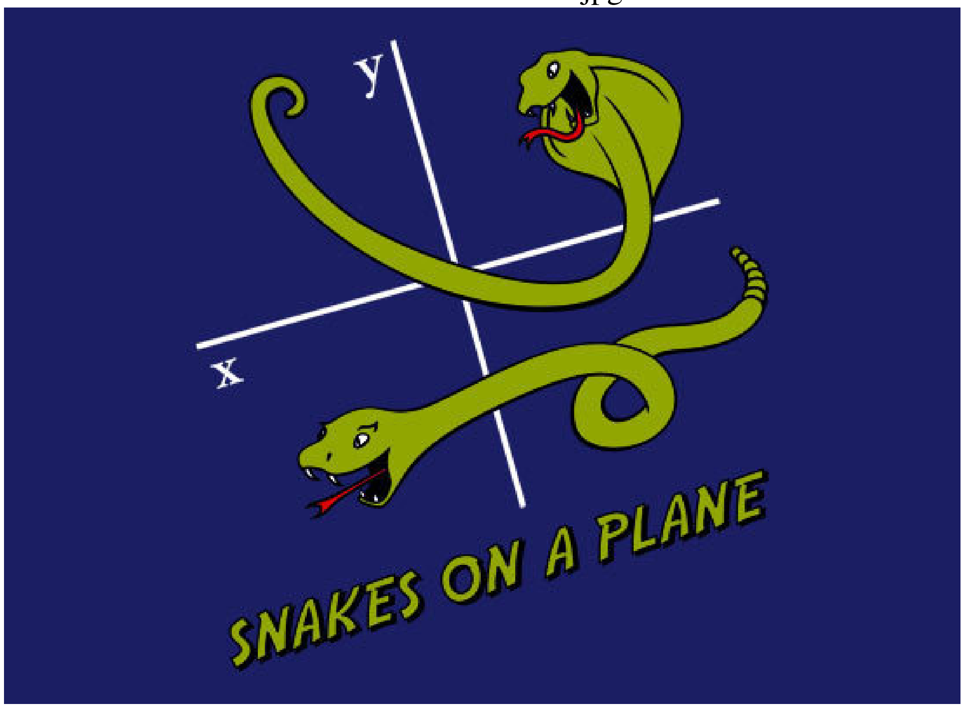
   Note: If you're completely stuck, then you can do problem 8 on page 12 instead, for a maximum of 12 points.

(This page is left blank in case you need more space to work on problem 7.)

8. (**ONLY** do this one if you got completely stuck on problem 7.)

If $12\ cm^3$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

1A/Practice Exams/Snake.jpg



Bonus 1 *(5 points)* Show that $f(x) = \ln(x) - x$ does not have a slant asymptote at ∞.

Hint: Assume $f(x)$ has a slant asymptote $y = mx + b$ at ∞. Calculate m, then calculate b, and find a contradiction!

Bonus 2 *(5 points)* Assume $-1 < f(x) < 1$ and $f'(x) \neq 1$ for all x. Show that f has exactly one fixed point.

Definition: a is a fixed point of f if $f(a) = a$

Hints:
At least one fixed point: Show that $g(x) = f(x) - x$ has at least one zero on $[-1, 1]$.

At most one fixed point: Assume f has 2 fixed points a and b, then $f(a) = a$ and $f(b) = b$, and find a contradiction!

Note: See the comments on the next page!

Discussion of Bonus 2:

Bonus 2 is part of a more general theorem, the **Brouwer fixed point theorem**. It states that any function f with domain $B(0,1)$ (the open ball of center 0 and radius 1 in $\mathbb{R}^n$) and range $B(0,1)$ has at least one fixed point. (in the previous problem, $B(0,1) = (-1,1)$). Moreover, if $f'(x) \neq 1$, then f has exactly one fixed point.

Here are some cool applications of this theorem:

(1) No matter how well you shake a snowglobe, then there will always be one snowflake which lands on exactly same position it started!
(2) If you stir a cocktail glass, then there is always one molecule which never changes position. And in most cases there is only one such molecule (unless you're rotating the glass).
(3) Suppose there is a hurricane in New York, and everyone gets sweeped to a different place. Then there is one lucky person who gets sweeped to the same place he/she started!
(4) Take an ordinary map of a country, and suppose that that map is laid out on a table inside that country. There will always be a 'You are Here" point on the map which represents that same point in the country.
(5) Have you ever looked at two mirrors that are across from each other? There seems to be an infinite number of smaller and smaller mirrors! However, there is always one point on all those mirrors which has always the same height (roughly at your belly button).
(6) Brouwer's fixed point theorem is used to prove the fundamental theorem of differential equations (namely that differential equations have solutions), as well as the implicit function theorem.

Any comments about this exam? (too long? too hard?)